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Boundedness and convergence to steady states in a two-species chemotaxis system with logistic source

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1. Introduction

We consider the two-species chemotaxis system

$$(1.1) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1-u), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1-v), & x \in \Omega, \ t > 0, \\ w_t = d\Delta w + h(u, v, w), & x \in \Omega, \ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$. The initial data u_0 , v_0 and w_0 are assumed to be nonnegative functions. The unknown functions $u(x, t)$ and $v(x, t)$ represent the population densities of two species and $w(x, t)$ shows the concentration of the substance at place x and time t .

In a mathematical view, global existence and behavior of solutions are fundamental theme. However, the problem (1.1) has some difficult points caused by the logistic term and by generalization of χ_i and h . For example, we cannot use the Lyapunov function. To overcome the difficulty, Negreanu–Tello [9, 10] built a technical way to prove global existence and asymptotic behavior of solutions to (1.1). In [10] they dealt with (1.1) when $d = 0$, $\mu_i > 0$ under the condition

$$\exists \bar{w} \geq w_0; \ h(\bar{u}, \bar{v}, \bar{w}) \leq 0,$$

where \bar{u} , \bar{v} satisfy some representations determined by \bar{w} . In [9] they studied (1.1) when $0 < d < 1$, $\mu_i = 0$ under similar conditions as in [10] and

$$(1.2) \quad \chi'_i + \frac{1}{1-d}\chi_i^2 \leq 0 \quad (i = 1, 2).$$

They supposed in [9, 10] that the functions h , χ_i for $i = 1, 2$ generalize of the prototypical case $\chi_i(w) = \frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ ($\chi_{0,i} > 0$, $\sigma_i \geq 1$), $h(u, v, w) = u + v - w$. As to the special case that $d = 1$ and $h(u, v, w) = u + v - w$, Zhang–Li [13] proved global existence of solutions to (1.1) under the assumption that μ_i is small and $\chi_i(w) \leq \frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ for $\sigma_i > 1$, $\chi_{0,i} > 0$ being small enough.

The purpose of the present paper is to obtain global existence and asymptotic stability of solutions to (1.1) without the restriction of $0 \leq d < 1$. We shall suppose throughout this paper that h, χ_i ($i = 1, 2$) satisfy the following conditions:

$$(1.3) \quad \chi_i \in C^{1+\theta}([0, \infty)) \cap L^1(0, \infty) \quad (0 < \exists \theta < 1), \quad \chi_i > 0 \quad (i = 1, 2),$$

$$(1.4) \quad h \in C^1([0, \infty) \times [0, \infty) \times [0, \infty)), \quad h(0, 0, 0) \geq 0,$$

$$(1.5) \quad \exists \gamma > 0; \quad \frac{\partial h}{\partial u}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial v}(u, v, w) \geq 0, \quad \frac{\partial h}{\partial w}(u, v, w) \leq -\gamma,$$

$$(1.6) \quad \exists \delta > 0, \exists M > 0; \quad |h(u, v, w) + \delta w| \leq M(u + v + 1),$$

$$(1.7) \quad \exists k_i > 0; \quad -\chi_i(w)h(0, 0, w) \leq k_i \quad (i = 1, 2).$$

We also assume that

$$(1.8) \quad \exists p > n; \quad 2d\chi'_i(w) + \left((d-1)p + \sqrt{(d-1)^2p^2 + 4dp} \right) [\chi_i(w)]^2 \leq 0 \quad (i = 1, 2).$$

The above conditions cover the prototypical example $\chi_i(w) = \frac{\chi_{0,i}}{(1+w)^{\sigma_i}}$ ($\chi_{0,i} > 0, \sigma_i > 1$), $h(u, v, w) = u + v - w$. We assume that the initial data u_0, v_0, w_0 satisfy

$$(1.9) \quad 0 \leq u_0 \in C(\bar{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\bar{\Omega}) \setminus \{0\}, \quad 0 \leq w_0 \in W^{1,q}(\Omega) \quad (\exists q > n).$$

Now the main results read as follows. The first theorem is concerned with global existence and boundedness in (1.1).

Theorem 1.1. *Let $d \geq 0, \mu_i > 0$ ($i = 1, 2$). Assume that h, χ_i satisfy (1.3)–(1.8). Then for any u_0, v_0, w_0 satisfying (1.9) for some $q > n$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$u, v, w \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \quad \text{when } d > 0,$$

$$u, v, w \in C([0, \infty); W^{1,q}(\Omega)) \cap C^1((0, \infty); W^{1,q}(\Omega)) \quad \text{when } d = 0,$$

which satisfy (1.1). Moreover, the solution (u, v, w) is uniformly bounded, i.e., there exists a constant $C_1 > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all } t \geq 0.$$

Remark 1.1. When $0 < d < 1$, we note that the condition (1.8) in Theorem 1.1 relaxes (1.2) assumed in [9], because the following relation holds:

$$\frac{(d-1)p + \sqrt{(d-1)^2p^2 + 4dp}}{2d} < \frac{1}{1-d}.$$

Now the second one, which gives asymptotic stability in (1.1), read as follows. We first introduce some notation. Since Theorem 1.1 guarantees that u, v and w exist globally and are bounded and nonnegative, it is possible to define nonnegative numbers α, β by

$$(1.10) \quad \alpha := \max_{(u,v,w) \in I} h_u(u, v, w), \quad \beta := \max_{(u,v,w) \in I} h_v(u, v, w),$$

where $I = (0, C_1)^3$ and C_1 is defined in Theorem 1.1.

Theorem 1.2. *Let $d > 0$, $\mu_i > 0$ ($i = 1, 2$). Under the conditions (1.3)–(1.9) and*

$$(1.11) \quad \alpha > 0, \quad \beta > 0, \quad \chi_1(0)^2 < \frac{16\mu_1 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta}, \quad \chi_2(0)^2 < \frac{16\mu_2 d\gamma}{\alpha^2 + \beta^2 + 2\alpha\beta},$$

the unique global solution (u, v, w) of (1.1) satisfies that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0),$$

where $\tilde{w} \geq 0$ such that $h(1, 1, \tilde{w}) = 0$.

Remark 1.2. From (1.4)–(1.6) there exists \tilde{w} such that $h(1, 1, \tilde{w}) = 0$. Indeed, if we choose $\bar{w} \geq 3M/\delta$, then (1.6) yields that $h(1, 1, \bar{w}) \leq 3M - \delta\bar{w} \leq 0$. On the other hand, (1.4) and (1.5) imply that $h(1, 1, 0) \geq h(0, 0, 0) \geq 0$. Hence, by the intermediate value theorem there exists $\tilde{w} \geq 0$ such that

$$h(1, 1, \tilde{w}) = 0.$$

The strategy for the proof of Theorem 1.1 is to construct estimates for $\int_\Omega u^p$ and $\int_\Omega v^p$. One of the keys for this strategy is to derive inequality

$$(1.12) \quad \frac{d}{dt} \int_\Omega u^p [f_1(w)]^{-r} \leq a \int_\Omega u^p [f_1(w)]^{-r} - b \left(\int_\Omega u^p [f_1(w)]^{-r} \right)^{\frac{p+1}{p}}$$

for some positive constants a, b , where

$$f_1(w) := \exp \left\{ \int_0^w \chi_1(s) ds \right\}.$$

Negreanu–Tello [9, 10] proved a similar differential inequality for “all” $p \geq 1$ and $r := \frac{(p-1)p}{p-d(p-1)}$. In this work we derive (1.12) for “some” $p > n$ and some $r = r(d, p) > 0$ by modifying the proof in [9, 10]. This enables us to improve the previous work and to remove the restriction of $0 \leq d < 1$. On the other hand, the strategy for the proof of Theorem 1.2 is to modify an argument in [8]. The key for this strategy is to construct the following energy estimate:

$$\frac{d}{dt} E(t) \leq -\varepsilon \left(\int_\Omega (u - 1)^2 + \int_\Omega (v - 1)^2 + \int_\Omega (w - \tilde{w})^2 \right)$$

with some function $E(t) \geq 0$ and some $\varepsilon > 0$. This strategy enables us to improve the conditions assumed in [7].

This paper is organized as follows. In Section 2 we collect basic facts which will be used later. In Section 3 we prove global existence and boundedness (Theorem 1.1). Section 4 is devoted to the proof of asymptotic stability (Theorem 1.2).

2. Preliminaries

In this paper we need the following well-known facts concerning the Laplacian in Ω supplemented with homogeneous Neumann boundary conditions (for details, see [4, 5]).

Lemma 2.1. *Suppose $k > 0$. Let Δ denote the realization of the Laplacian in $L^s(\Omega)$ with domain $\{z \in W^{2,s}(\Omega) \mid \nabla z \cdot \nu = 0 \text{ on } \partial\Omega\}$ for $s \in (1, \infty)$. Then the operator $-\Delta + k$ is sectorial and possesses closed fractional powers $(-\Delta + k)^\eta$, $\eta \in (0, 1)$, with dense domain $D((-\Delta + k)^\eta)$. Moreover, the following holds.*

- (i) *If $m \in \{0, 1\}$, $p \in [1, \infty]$ and $q \in (1, \infty)$, then there exists a constant $c_1 > 0$ such that for all $z \in D((-\Delta + k)^\eta)$,*

$$\|z\|_{W^{m,p}(\Omega)} \leq c_1 \|(-\Delta + k)^\eta z\|_{L^q(\Omega)},$$

provided that $m < 2\eta$ and $m - n/p < 2\eta - n/q$.

- (ii) *Suppose $p \in [1, \infty)$. Then the associated heat semigroup $(e^{t\Delta})_{t \geq 0}$ maps $L^p(\Omega)$ into $D((-\Delta + k)^\eta)$ in any of the space $L^q(\Omega)$, $q \geq p$, and there exist $c_2 > 0$ and $\lambda > 0$ such that for all $z \in L^p(\Omega)$,*

$$\|(-\Delta + k)^\eta e^{t\Delta} z\|_{L^q(\Omega)} \leq c_2 t^{-\eta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda t} \|z\|_{L^p(\Omega)} \quad (t > 0).$$

- (iii) *Let $p \in (1, \infty)$. Then there exists $\lambda > 0$ such that for every $\varepsilon > 0$ there exists $c_3 > 0$ such that for all \mathbb{R}^n -valued $\omega \in C_0^\infty(\Omega)$,*

$$(2.1) \quad \|(-\Delta + k)^\eta e^{t\Delta} \nabla \cdot \omega\|_{L^p(\Omega)} \leq c_3 t^{-\eta - \varepsilon - \frac{1}{2}} e^{-\lambda t} \|\omega\|_{L^p(\Omega)} \quad (t > 0).$$

Accordingly, the operator $(-\Delta + k)^\eta e^{t\Delta} \nabla \cdot$ admits a unique extension to all of $L^p(\Omega)$ which, again denoted by $(-\Delta + k)^\eta e^{t\Delta} \nabla \cdot$, satisfies (2.1) for all \mathbb{R}^n -valued $w \in L^p(\Omega)$.

Lemma 2.2. *Let $d \geq 0$, $\mu_i \geq 0$ ($i = 1, 2$). Assume that h, χ_i satisfy (1.3), (1.4), (1.6). Then for any u_0, v_0, w_0 satisfying (1.9) for some $q > n$, there exist $T_{\max} \in (0, \infty]$ and an exactly one pair (u, v, w) of nonnegative functions*

$$\begin{aligned} u, v, w &\in C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \quad \text{when } d > 0, \\ u, v, w &\in C([0, T_{\max}); W^{1,q}(\Omega)) \cap C^1((0, T_{\max}); W^{1,q}(\Omega)) \quad \text{when } d = 0, \end{aligned}$$

which satisfy (1.1). Moreover,

$$\text{either } T_{\max} = \infty \quad \text{or} \quad \lim_{t \rightarrow T_{\max}} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}) = \infty.$$

Proof. We first consider the case $d > 0$. The proof of local existence of classical solutions to (1.1) is based on a standard contraction mapping argument, which can be found in [11, 12]. The case $d = 0$ is show in [10]. Finally the maximum principle is applied to yield $u > 0$, $v > 0$, $w \geq 0$ in $\Omega \times (0, T_{\max})$. \square

3. Global existence and boundedness

Let (u, v, w) be the solution to (1.1) on $[0, T_{\max})$ as in Lemma 2.2. We introduce the functions $f_1 = f_1(w)$ and $f_2 = f_2(w)$ by

$$(3.1) \quad f_i(w) := \exp \left\{ \int_0^w \chi_i(s) ds \right\} \quad \text{for } i = 1, 2$$

to prove the following lemma.

Lemma 3.1. *Let $d \geq 0$, $\mu_i \geq 0$ ($i = 1, 2$). Assume that χ_i satisfy (1.3) and (1.8) with some $p > n$. Then there exists $r = r(d, p) > 0$ such that*

$$(3.2) \quad \frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq p\mu_1 \int_{\Omega} u^p f_1^{-r} (1-u) - r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w),$$

$$(3.3) \quad \frac{d}{dt} \int_{\Omega} v^p f_2^{-r} \leq p\mu_2 \int_{\Omega} v^p f_2^{-r} (1-v) - r \int_{\Omega} v^p f_2^{-r} \chi_2(w) h(u, v, w).$$

Proof. We let $p \geq 1$ be fixed later. From the first and third equations in (1.1) we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p f_1^{-r} &= p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot (\nabla u - u \chi_1(w) \nabla w) + p\mu_1 \int_{\Omega} u^p f_1^{-r} (1-u) \\ &\quad - rd \int_{\Omega} u^p f_1^{-r} \chi_1(w) \Delta w - r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w). \end{aligned}$$

Denoting by I_1 and I_2 the first and third terms on the right-hand side as

$$I_1 := p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot (\nabla u - u \chi_1(w) \nabla w),$$

$$I_2 := -rd \int_{\Omega} u^p f_1^{-r} \chi_1(w) \Delta w,$$

we can write as

$$(3.4) \quad \frac{d}{dt} \int_{\Omega} u^p f_1^{-r} = I_1 + I_2 + p\mu_1 \int_{\Omega} u^p f_1^{-r} (1-u) - r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w).$$

We shall show that the following inequality:

$$\exists p > n, \exists r > 0; \quad I_1 + I_2 \leq 0.$$

Noting that

$$f_1 \nabla \left(\frac{u}{f_1} \right) = \nabla u - u \chi_1(w) \nabla w,$$

we obtain

$$\begin{aligned} I_1 &= p \int_{\Omega} u^{p-1} f_1^{-r} \nabla \cdot \left(f_1 \nabla \left(\frac{u}{f_1} \right) \right) \\ &= p \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-1} f_1^{-r+p-1} \nabla \cdot \left(f_1 \nabla \left(\frac{u}{f_1} \right) \right) \\ &= -p(p-1) \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left(\frac{u}{f_1} \right) \right|^2 \\ &\quad - p(-r+p-1) \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left(\frac{u}{f_1} \right) \cdot \nabla w. \end{aligned}$$

Similarly, we see that

$$\begin{aligned}
 I_2 &= -rd \int_{\Omega} \left(\frac{u}{f_1} \right)^p f_1^{-r+p} \chi_1(w) \Delta w \\
 &= rdp \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left(\frac{u}{f_1} \right) \cdot \nabla w \\
 &\quad + rd \int_{\Omega} \left(\frac{u}{f_1} \right)^p f_1^{-r+p} ((-r+p)[\chi_1(w)]^2 + \chi_1'(w)) |\nabla w|^2.
 \end{aligned}$$

Therefore it follows that

$$\begin{aligned}
 I_1 + I_2 &= -p(p-1) \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left(\frac{u}{f_1} \right) \right|^2 \\
 &\quad - (p(p-1) - (1+d)pr) \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-1} f_1^{-r+p} \chi_1(w) \nabla \left(\frac{u}{f_1} \right) \cdot \nabla w \\
 &\quad + \int_{\Omega} \left(\frac{u}{f_1} \right)^p f_1^{-r+p} (dr(-r+p)[\chi_1(w)]^2 + dr\chi_1'(w)) |\nabla w|^2 \\
 &= -p(p-1) \int_{\Omega} \left(\frac{u}{f_1} \right)^{p-2} f_1^{-r+p} \left| \nabla \left(\frac{u}{f_1} \right) \right|^2 + \frac{p(p-1) - (1+d)pr}{2p(p-1)} \chi_1(w) \frac{u}{f_1} |\nabla w|^2 \\
 &\quad + \int_{\Omega} \left(\frac{u}{f_1} \right)^p f_1^{-r+p} \left[\left(\frac{(p(p-1) - (1+d)pr)^2}{4p(p-1)} + dr(-r+p) \right) [\chi_1(w)]^2 + dr\chi_1'(w) \right] |\nabla w|^2.
 \end{aligned}$$

Here we write as

$$\begin{aligned}
 &\left(\frac{(p(p-1) - (1+d)pr)^2}{4p(p-1)} + dr(-r+p) \right) [\chi_1(w)]^2 + dr\chi_1'(w) \\
 &= \frac{1}{4p(p-1)} (a_1 r^2 + 2a_2 r + a_3),
 \end{aligned}$$

where a_1, a_2, a_3 are given by

$$\begin{aligned}
 a_1 &:= ((d-1)^2 p + 4d) [\chi_1(w)]^2, \\
 a_2 &:= (p-1) (p(d-1) [\chi_1(w)]^2 + 2d\chi_1'(w)), \\
 a_3 &:= p(p-1)^2 [\chi_1(w)]^2.
 \end{aligned}$$

Then there exists $p > n$ such that the discriminant

$$D_r = 4(p-1)^2 [(p\chi_1^2(d-1) + 2d\chi_1')^2 - p\chi_1^4(p(d-1)^2 + 4d)]$$

is nonnegative in view of (1.8). Therefore we have that there exists $r > 0$ such that

$$I_1 + I_2 \leq 0.$$

Hence (3.4) implies

$$\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq p\mu_1 \int_{\Omega} u^p f_1^{-r} (1-u) - r \int_{\Omega} u^p f_1^{-r} \chi_1 h(u, v, w).$$

This means that (3.2) holds. In the same way, we obtain (3.3). \square

Lemma 3.2. Let $d \geq 0$, $\mu_i > 0$ ($i = 1, 2$). Assume that h, χ_i satisfy (1.3)–(1.5), (1.7), and (1.8) with some positive constants k_i ($i = 1, 2$) and $p > n$, then

$$(3.5) \quad \|u(t)\|_{L^p(\Omega)} \leq \left(e^{\|\chi_1\|_{L^1(0,\infty)}} \right)^{r/p} \max \left\{ \|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + rk_1}{p\mu_1} |\Omega|^{1/p} \right\},$$

$$(3.6) \quad \|v(t)\|_{L^p(\Omega)} \leq \left(e^{\|\chi_2\|_{L^1(0,\infty)}} \right)^{r/p} \max \left\{ \|v_0\|_{L^p(\Omega)}, \frac{p\mu_2 + rk_2}{p\mu_2} |\Omega|^{1/p} \right\}.$$

Proof. From the mean value theorem, the condition (1.5) and the fact that $u, v > 0$, it follows that for some ξ_1, ξ_2 satisfying $0 \leq \xi_1 \leq u$ and $0 \leq \xi_2 \leq v$,

$$\begin{aligned} h(u, v, w) &= \frac{\partial h}{\partial u}(\xi_1, v, w)u + \frac{\partial h}{\partial v}(0, \xi_2, w)v + h(0, 0, w) \\ &\geq h(0, 0, w). \end{aligned}$$

This together with the condition (1.7) leads to

$$\begin{aligned} (3.7) \quad -r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(u, v, w) &\leq -r \int_{\Omega} u^p f_1^{-r} \chi_1(w) h(0, 0, w) \\ &\leq k_1 r \int_{\Omega} u^p f_1^{-r}. \end{aligned}$$

Combining (3.2) with (3.7), we obtain

$$\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq (\mu_1 p + k_1 r) \int_{\Omega} u^p f_1^{-r} - \mu_1 p \int_{\Omega} u^{p+1} f_1^{-r}.$$

Hence the Hölder inequality gives

$$\frac{d}{dt} \int_{\Omega} u^p f_1^{-r} \leq (\mu_1 p + k_1 r) \int_{\Omega} u^p f_1^{-r} - \mu_1 p |\Omega|^{-1/p} \left(\int_{\Omega} u^p f_1^{-r} \right)^{(p+1)/p}.$$

Solving this differential inequality, we infer

$$\left(\int_{\Omega} u^p f_1^{-r} \right)^{1/p} \leq \max \left\{ \left(\int_{\Omega} u_0^p f_1^{-r} \right)^{1/p}, \frac{p\mu_1 + rk_1}{p\mu_1} |\Omega|^{1/p} \right\}.$$

Recalling the definition (3.1), we notice the relation $1 \leq f_1(w) \leq e^{\|\chi_1\|_{L^1(0,\infty)}}$, which yields (3.5). In the same way, we obtain (3.6). \square

Remark 3.1. When $d = 0$, (3.2), (3.3), (3.5) and (3.6) still hold for all $p \geq 1$. Indeed, we have only to choose $r = 1 - p$ in the above proof.

Proof of Theorem 1.1. First consider the case $d > 0$. We let $\tau \in (0, T_{\max})$. In view of Lemma 2.2 it is sufficient to make sure that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_u(\tau), \quad \|v(t)\|_{L^\infty(\Omega)} \leq C_v(\tau), \quad \|w(t)\|_{L^\infty(\Omega)} \leq C_w(\tau), \quad t \in (\tau, T_{\max})$$

holds with some $C_u(\tau)$, $C_v(\tau)$, $C_w(\tau) > 0$. We let $\rho \in \left(\frac{p+n}{2p}, 1\right)$. This means $1 < 2\rho - \frac{n}{p}$. Writing as

$$w_t = d(\Delta - \delta/d)w + h(u, v, w) + \delta w,$$

and applying the variation of constants formula for w , we have

$$w(t) = e^{dt(\Delta - \delta/d)}w_0 + \int_0^t e^{d(t-s)(\Delta - \delta/d)}(h(u(s), v(s), w(s)) + \delta w(s)) ds.$$

From Lemma 2.1 and (1.6) we obtain that for all $t \in (\tau, T_{\max})$,

$$\begin{aligned} \|w(t)\|_{W^{1,\infty}(\Omega)} &\leq c_1 \|(-\Delta + \delta/d)^\rho w(t)\|_{L^p(\Omega)} \\ &\leq c_1 c_2 t^{-\rho} e^{-\lambda t} \|w_0\|_{L^p(\Omega)} \\ &\quad + c_1 c_2 \int_0^t (t-s)^{-\rho} e^{-\lambda(t-s)} \|h(u(s), v(s), w(s)) + \delta w(s)\|_{L^p(\Omega)} ds \\ &\leq c_1 c_2 \tau^{-\rho} e^{-\lambda \tau} \|w_0\|_{L^p(\Omega)} + c_1 c_2 c_4 \int_0^t (t-s)^{-\rho} e^{-\lambda(t-s)} ds, \end{aligned}$$

where $c_4 := \sup_{0 \leq s < T_{\max}} \{M(\|u(s)\|_{L^p(\Omega)} + \|v(s)\|_{L^p(\Omega)} + 1)\}$ ($< \infty$ by Lemma 3.2). Noting that

$$\int_0^t (t-s)^{-\rho} e^{-\lambda(t-s)} ds \leq \int_0^\infty r^{-\rho} e^{-\lambda r} dr < \infty,$$

we deduce that

$$(3.8) \quad \|w(t)\|_{W^{1,\infty}(\Omega)} \leq c_1 c_2 \left(\tau^{-\rho} e^{-\lambda \tau} + c_4 \int_0^\infty r^{-\rho} e^{-\lambda r} dr \right) =: C_w(\tau).$$

Since (1.8) implies $\chi'_1 < 0$, it follows from (3.5) and (3.8) that for all $t \in (\tau/2, T_{\max})$,

$$(3.9) \quad \begin{aligned} \|u(t)\chi_1(w(t))\nabla w(t)\|_{L^p(\Omega)} &\leq \chi_1(0)\|u(t)\|_{L^p(\Omega)}\|\nabla w(t)\|_{L^\infty(\Omega)} \\ &\leq \chi_1(0) \sup_{0 \leq t < T_{\max}} \|u(t)\|_{L^p(\Omega)} C_w(\tau/2) =: c_5. \end{aligned}$$

Employing the variation of constants formula for u yields

$$\begin{aligned} u(t) &= e^{(t-\tau/2)(\Delta-1)}u\left(\frac{\tau}{2}\right) - \int_{\tau/2}^t e^{(t-s)(\Delta-1)}\nabla \cdot (u(s)\chi_1(w(s))\nabla w(s)) ds \\ &\quad + \int_{\tau/2}^t e^{(t-s)(\Delta-1)}[(\mu_1+1)u(s) - \mu_1 u(s)^2] ds \\ &=: J_1 + J_2 + J_3, \quad t \in (\tau, T_{\max}). \end{aligned}$$

Let $\eta \in \left(\frac{n}{2p}, \frac{1}{2}\right)$ and $\varepsilon \in (0, \frac{1}{2} - \eta)$. Then we observe that $0 < 2\eta - \frac{n}{p}$ and $\eta + \varepsilon + \frac{1}{2} < 1$. By Lemmas 2.1 and 3.2 we see that for all $t \in (\tau, T_{\max})$,

$$\begin{aligned} \|J_1\|_{L^\infty(\Omega)} &= \left\| e^{(t-\tau/2)(\Delta-1)} u \left(\frac{\tau}{2} \right) \right\|_{L^\infty(\Omega)} \\ &\leq c_1 \left\| (-\Delta + 1)^\eta e^{(t-\tau/2)(\Delta-1)} u \left(\frac{\tau}{2} \right) \right\|_{L^p(\Omega)} \\ &\leq c_1 c_2 \left(t - \frac{\tau}{2} \right)^{-\eta} e^{-\lambda t} \left\| u \left(\frac{\tau}{2} \right) \right\|_{L^p(\Omega)} \\ &\leq 2^\eta c_1 c_2 \tau^{-\eta} e^{-\eta\tau} \sup_{0 \leq t < T_{\max}} \|u(t)\|_{L^p(\Omega)}. \end{aligned}$$

Using Lemma 2.1 and (3.9), we obtain

$$\begin{aligned} \|J_2\|_{L^\infty(\Omega)} &\leq \int_{\tau/2}^t \|e^{(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s))\|_{L^\infty(\Omega)} ds \\ &\leq c_1 \int_{\tau/2}^t \|(-\Delta + 1)^\eta e^{(t-s)(\Delta-1)} \nabla \cdot (u(s)\chi_1(w(s))\nabla w(s))\|_{L^p(\Omega)} ds \\ &\leq c_1 c_3 \int_{\tau/2}^t (t-s)^{-\eta-\varepsilon-1/2} e^{-(\nu+1)(t-s)} \|u(s)\chi_1(w(s))\nabla w(s)\|_{L^p(\Omega)} ds \\ &\leq c_1 c_3 c_5 \int_0^\infty r^{-(\eta+\varepsilon+1/2)} e^{-(\nu+1)r} dr. \end{aligned}$$

Since the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ has the order preserving property, we infer

$$\begin{aligned} J_3 &= \int_{\tau/2}^t e^{(t-s)(\Delta-1)} \left[-\mu_1 \left(u(s) - \frac{\mu_1 + 1}{2\mu_1} \right)^2 + \frac{(\mu_1 + 1)^2}{4\mu_1} \right] ds \\ &\leq \frac{(\mu_1 + 1)^2}{4\mu_1} \int_{\tau/2}^t e^{(t-s)\Delta} e^{-(t-s)} ds, \end{aligned}$$

and moreover, by the maximum principle we have

$$\begin{aligned} J_3 &\leq \frac{(\mu_1 + 1)^2}{4\mu_1} \int_{\tau/2}^t e^{-(t-s)} ds \\ &\leq \frac{(\mu_1 + 1)^2}{4\mu_1} (1 - e^{-\tau/2}). \end{aligned}$$

Therefore we obtain that there exists $C_u(\tau) > 0$ such that

$$\begin{aligned} u(t) &\leq \|J_1\|_{L^\infty(\Omega)} + \|J_2\|_{L^\infty(\Omega)} + J_3 \\ &\leq C_u(\tau), \quad t \in (\tau, T_{\max}). \end{aligned}$$

The positivity of u yields that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C_u(\tau), \quad t \in (\tau, T_{\max}).$$

The same argument as for u gives the $L^\infty(\Omega)$ bound for v . This completes the proof in the case $d > 0$.

Next consider the case $d = 0$. From Remark 3.1 we have

$$\|u(t)\|_{L^p(\Omega)} \leq \exp\{\|\chi_1\|_{L^1(0,\infty)}\}^{(p-1)/p} \max\left\{\|u_0\|_{L^p(\Omega)}, \frac{p\mu_1 + (p-1)k_1}{p\mu_1}|\Omega|^{1/p}\right\}$$

for all $p \geq 1$. Taking the limits as $p \rightarrow \infty$, we obtain the $L^\infty(\Omega)$ bound for u , and similarly for v . The L^∞ bound for w follows from

$$w(t) = e^{-\delta t}w_0 + \int_0^t e^{-\delta(t-s)}(h(u, v, w) + \delta w) ds.$$

This completes the proof when $d = 0$. \square

4. Asymptotic behavior

In this section we will establish asymptotic stability of solutions to (1.1). For the proof of Theorem 1.2, we shall prepare some elementary results.

Lemma 4.1 ([1, Lemma 3.1]). *Suppose that $f : (1, \infty) \rightarrow \mathbb{R}$ is a uniformly continuous nonnegative function satisfying $\int_1^\infty f(t) dt < \infty$. Then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Lemma 4.2. *Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$. Suppose that*

$$(4.1) \quad a_1 > 0, \quad a_3 > 0, \quad a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3} > 0.$$

Then

$$(4.2) \quad a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 \geq 0$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. From straightforward calculations we obtain

$$\begin{aligned} & a_1x^2 + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ &= a_1 \left(x + \frac{a_2z}{2a_1}\right)^2 + a_3 \left(y + \frac{a_4z}{2a_3}\right)^2 + \left(a_5 - \frac{a_2^2}{4a_1} - \frac{a_4^2}{4a_3}\right)z^2. \end{aligned}$$

In view of the above equation, (4.1) leads to (4.2). \square

Now we will prove the key estimate for the proof of Theorem 1.2.

Lemma 4.3. *Let (u, v, w) be a solution to (1.1). Under the conditions (1.3)–(1.9) and (1.11), there exist $\delta_1, \delta_2 > 0$ and $\varepsilon > 0$ such that the nonnegative functions E_1 and F_1 defined by*

$$E_1(t) := \int_{\Omega} (u - 1 - \log u) + \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} (v - 1 - \log v) + \frac{\delta_2}{2} \int_{\Omega} (w - \tilde{w})^2$$

and

$$F_1(t) := \int_{\Omega} (u-1)^2 + \int_{\Omega} (v-1)^2 + \int_{\Omega} (w-\tilde{w})^2$$

satisfy

$$(4.3) \quad \frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \quad (t > 0).$$

Proof. Thanks to (1.11), we can choose $\delta_1 = \frac{\beta}{\alpha} > 0$ and $\delta_2 > 0$ satisfying

$$(4.4) \quad \max \left\{ \frac{\chi_1(0)^2(1+\delta_1)}{4d}, \frac{\mu_1\chi_2(0)^2(1+\delta_1)}{4\mu_2d} \right\} < \delta_2 < \frac{4\mu_1\gamma\delta_1}{\alpha^2\delta_1 + \beta^2}.$$

We denote by $A_1(t)$, $B_1(t)$, $C_1(t)$ the functions defined as

$$\begin{aligned} A_1(t) &:= \int_{\Omega} (u-1-\log u), \quad B_1(t) = \int_{\Omega} (v-1-\log v), \\ C_1(t) &:= \frac{1}{2} \int_{\Omega} (w-\tilde{w})^2, \end{aligned}$$

and we write as

$$E_1(t) = A_1(t) + \delta_1 \frac{\mu_1}{\mu_2} B_1(t) + \delta_2 C_1(t).$$

The Taylor formula applied to $H(s) = s - \log s$ ($s \geq 0$) yields $A_1(t) = \int_{\Omega} (H(u) - H(1))$ is a nonnegative function for $t > 0$ (more detail, see [1, Lemma 3.2]). Similarly, we have that $B_1(t)$ is a positive function. By straightforward calculations we infer

$$\begin{aligned} \frac{d}{dt} A_1(t) &= -\mu_1 \int_{\Omega} (u-1)^2 - \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w, \\ \frac{d}{dt} B_1(t) &= -\mu_2 \int_{\Omega} (v-1)^2 - \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w, \\ \frac{d}{dt} C_1(t) &= \int_{\Omega} h_u(u-1)(w-\tilde{w}) + \int_{\Omega} h_v(v-1)(w-\tilde{w}) + \int_{\Omega} h_w(w-\tilde{w})^2 \\ &\quad - d \int_{\Omega} |\nabla w|^2 \end{aligned}$$

with some derivatives h_u , h_v and h_w . Hence we have

$$(4.5) \quad \frac{d}{dt} E_1(t) = I_3(t) + I_4(t),$$

where

$$\begin{aligned} I_3(t) &:= -\mu_1 \int_{\Omega} (u-1)^2 - \delta_1 \mu_1 \int_{\Omega} (v-1)^2 + \delta_2 \int_{\Omega} h_u(u-1)(w-\tilde{w}) \\ &\quad + \delta_2 \int_{\Omega} h_v(v-1)(w-\tilde{w}) + \delta_2 \int_{\Omega} h_w(w-\tilde{w})^2 \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} I_4(t) := & - \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w - \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \\ & + \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w - d\delta_2 \int_{\Omega} |\nabla w|^2. \end{aligned}$$

At first, we shall show from Lemma 4.2 that there exists $\varepsilon_1 > 0$ such that

$$(4.7) \quad I_3(t) \leq -\varepsilon_1 \left(\int_{\Omega} (u-1)^2 + \int_{\Omega} (v-1)^2 + \int_{\Omega} (w-\tilde{w})^2 \right).$$

To see this, we put

$$\begin{aligned} g_1(\varepsilon) &:= \mu_1 - \varepsilon, & g_2(\varepsilon) &:= \delta_1 \mu_1 - \varepsilon, \\ g_3(\varepsilon) &:= (-\delta_2 h_w - \varepsilon) - \frac{h_u^2}{4(\mu_1 - \varepsilon)} \delta_2^2 - \frac{h_v^2}{4(\delta_1 \mu_1 - \varepsilon)} \delta_2^2. \end{aligned}$$

Since $\mu_1 > 0$ and $\delta_1 = \frac{\beta}{\alpha} > 0$, we have $g_1(0) = \mu_1 > 0$ and $g_2(0) = \delta_1 \mu_1 > 0$. In light of (1.5) and the definitions of $\delta_2, \alpha, \beta > 0$ (see (1.10) and (4.4)) we obtain

$$\begin{aligned} g_3(0) &= \delta_2 \left(-h_w - \left(\frac{h_u^2}{4\mu_1} + \frac{h_v^2}{4\delta_1 \mu_1} \right) \delta_2 \right) \\ &\geq \delta_2 \left(\gamma - \left(\frac{\alpha^2}{4\mu_1} + \frac{\beta^2}{4\delta_1 \mu_1} \right) \delta_2 \right) \\ &\geq \delta_2 \left(\gamma - \left(\frac{\alpha^2 \delta_1 + \beta}{4\delta_1 \mu_1} \right) \delta_2 \right) > 0. \end{aligned}$$

Combination of the above inequalities and the continuity of g_i for $i = 1, 2, 3$ yield that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thanks to Lemma 4.2 with

$$\begin{aligned} a_1 &= \mu_1 - \varepsilon_1, & a_2 &= -\delta_2 h_u, & a_3 &= \delta_1 \mu_1 - \varepsilon_1, \\ a_4 &= -\delta_2 h_v, & a_5 &= -\delta_2 h_w - \varepsilon_1, \\ x &= u(t) - 1, & y &= v(t) - 1, & z &= w(t) - \tilde{w}, \end{aligned}$$

we obtain (4.7) with $\varepsilon_1 > 0$. Lastly we will prove

$$(4.8) \quad I_4(t) \leq 0.$$

Noting that $\chi'_i < 0$ (from (1.8)) and then using the Young inequality, we have

$$\begin{aligned} \int_{\Omega} \frac{\chi_1(w)}{u} \nabla u \cdot \nabla w &\leq \chi_1(0) \int_{\Omega} \frac{|\nabla u \cdot \nabla w|}{u} \\ &\leq \frac{\chi_1(0)^2 (1 + \delta_1)}{4d\delta_2} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{d\delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2 \end{aligned}$$

and

$$\begin{aligned} \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{\chi_2(w)}{v} \nabla v \cdot \nabla w &\leq \chi_2(0) \delta_1 \frac{\mu_1}{\mu_2} \int_{\Omega} \frac{|\nabla v \cdot \nabla w|}{v} \\ &\leq \frac{\chi_2(0)^2 \delta_1 (1 + \delta_1)}{4d\delta_2} \left(\frac{\mu_1}{\mu_2} \right)^2 \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \frac{d\delta_1\delta_2}{1 + \delta_1} \int_{\Omega} |\nabla w|^2. \end{aligned}$$

Plugging these into (4.6) we infer

$$\begin{aligned} I_4(t) &\leq - \left(1 - \frac{\chi_1(0)^2 (1 + \delta_1)}{4d\delta_2} \right) \int_{\Omega} \frac{|\nabla u|^2}{u^2} \\ &\quad - \delta_1 \frac{\mu_1}{\mu_2} \left(1 - \frac{\mu_1 \chi_2(0)^2 (1 + \delta_1)}{4d\mu_2\delta_2} \right) \int_{\Omega} \frac{|\nabla v|^2}{v^2}. \end{aligned}$$

We note from the definition of $\delta_2 > 0$ that

$$\begin{aligned} 1 - \frac{\chi_1(0)^2 (1 + \delta_1)}{4d\delta_2} &> 0, \\ 1 - \frac{\mu_1 \chi_2(0)^2 (1 + \delta_1)}{4d\mu_2\delta_2} &> 0. \end{aligned}$$

Thus we have (4.8). Combination of (4.5), (4.7) and (4.8) implies the end of the proof. \square

Lemma 4.4. *Let (u, v, w) be a solution to (1.1). Under the conditions (1.3)–(1.9) and (1.11), (u, v, w) has the following asymptotic behavior:*

$$\|u(t) - 1\|_{L^\infty(\Omega)} \rightarrow 0, \quad \|v(t) - 1\|_{L^\infty(\Omega)} \rightarrow 0, \quad \|w(t) - \tilde{w}\|_{L^\infty(\Omega)} \rightarrow 0 \quad (t \rightarrow \infty).$$

Proof. Firstly the boundedness of $u, v, \nabla w$ and a standard parabolic regularity theory ([6]) yield that there exist $\theta \in (0, 1)$ and $C > 0$ such that

$$\|u\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times [1, t])} + \|v\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times [1, t])} + \|w\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times [1, t])} \leq C \quad \text{for all } t \geq 1.$$

Therefore in view of the Gagliardo–Nirenberg inequality

$$(4.9) \quad \|\varphi\|_{L^\infty(\Omega)} \leq c \|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\varphi\|_{L^2(\Omega)}^{\frac{2}{n+2}} \quad (\varphi \in W^{1,\infty}(\Omega)),$$

it is sufficient to show that

$$\|u(t) - 1\|_{L^2(\Omega)} \rightarrow 0, \quad \|v(t) - 1\|_{L^2(\Omega)} \rightarrow 0, \quad \|w(t) - \tilde{w}\|_{L^2(\Omega)} \rightarrow 0 \quad (t \rightarrow \infty).$$

We let

$$f_1(t) := \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 + \int_{\Omega} (w - \tilde{w})^2.$$

We have that $f_1(t)$ is a nonnegative function, and thanks to the regularity of u, v, w we can see that $f_1(t)$ is uniformly continuous. Moreover, integrating (4.3) over $(1, \infty)$, we infer from the positivity of $E_1(t)$ that

$$\int_1^\infty f_1(t) dt \leq \frac{1}{\varepsilon} E_1(1) < \infty.$$

Therefore we conclude from Lemma 4.1 that $f_1(t) \rightarrow 0$ ($t \rightarrow \infty$), which means

$$\int_{\Omega} (u-1)^2 + \int_{\Omega} (v-1)^2 + \int_{\Omega} (w-\tilde{w})^2 \rightarrow 0 \quad (t \rightarrow \infty).$$

This implies the end of the proof. \square

Lemma 4.5. *Let (u, v, w) be a solution to (1.1). Under the conditions (1.3)–(1.9) and (1.11), there exist $C > 0$ and $\lambda > 0$ such that*

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0).$$

Proof. From the L'Hôpital theorem applied to $H_1(s) := s - \log s$ we can see

$$(4.10) \quad \lim_{s \rightarrow 1} \frac{H_1(s) - H_1(1)}{(s-1)^2} = \lim_{s \rightarrow 1} \frac{H_1''(s)}{2} = \frac{1}{2}.$$

In view of the combination of (4.10) and $\|u - 1\|_{L^\infty(\Omega)} \rightarrow 0$ from Lemma 4.4 we obtain that there exists $t_0 > 0$ such that

$$(4.11) \quad \frac{1}{4} \int_{\Omega} (u-1)^2 \leq A_1(t) = \int_{\Omega} (H(u) - H(1)) \leq \int_{\Omega} (u-1)^2 \quad (t > t_0).$$

A similar argument yields that there exists $t_1 > t_0$ such that

$$(4.12) \quad \frac{1}{4} \int_{\Omega} (v-1)^2 \leq B_1(t) \leq \int_{\Omega} (v-1)^2 \quad (t > t_1).$$

We infer from (4.11) and the definitions of $E_1(t)$, $F_1(t)$ that

$$E_1(t) \leq c_6 F_1(t)$$

for all $t > t_1$ with some $c_6 > 0$. Plugging this into (4.3), we have

$$\frac{d}{dt} E_1(t) \leq -\varepsilon F_1(t) \leq -\frac{\varepsilon}{c_6} E_1(t) \quad (t > t_1),$$

which implies that there exist $c_7 > 0$ and $\ell > 0$ such that

$$E_1(t) \leq c_7 e^{-\ell t} \quad (t > t_1).$$

Thus we obtain from (4.11) and (4.12) that

$$\int_{\Omega} (u-1)^2 + \int_{\Omega} (v-1)^2 + \int_{\Omega} (w-\tilde{w})^2 \leq c_8 E_1(t) \leq c_7 c_8 e^{-\ell t}$$

for all $t > t_1$ with some $c_8 > 0$. From the Gagliardo–Nirenberg inequality (4.9) with the regularity of u, v, w , we achieve that there exist $C > 0$ and $\lambda > 0$ such that

$$\|u(t) - 1\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \|w(t) - \tilde{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad (t > 0).$$

This completes the proof of Lemma 4.5. \square

Proof of Theorem 1.2. Theorem 1.2 follows directly from Lemma 4.5. \square

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